Worksheet 6-4: Investigating Exponential Functions

1. Paper-folding Investigation: Exponential Growth *(Increasing)*

(a) Take a large rectangular sheet of paper and fold it in half. Unfold it and count the number of rectangles formed by the crease. Record the number of folds and the number of rectangles in the table provided in part (c), and then refold the paper.

(b) Fold the paper in half again. Unfold and record the number of rectangles formed by the creases. Refold the paper again.

(c) Continue folding in half and recording the number of rectangles until you can no longer fold the paper. Record your findings in the following table.

Number of Folds	Number of Rectangles	Common Ratio
0	1 $\times 2$	$2 \div 1 = 2$
1	2	$4 \div 2 = 2$
2	4	$8 \div 4 = 2$
3	8	$16 \div 8 = 2$
4	16	$32 \div 16 = 2$
5	32	

$y = 2^x$

(d) Graph the relation, showing the number of folds on the horizontal axis and the number of rectangles on the vertical axis. (Graph on next page.)

(e) If x represents the number of folds and y represents the number of rectangles, write an equation for the relation.

For exponential growth,

$y = b^x$, **where** $b > 0$.

$\boxed{x^0 = 1}$

(f) Complete the following table for your graph.

	Domain	Range	x-intercept	y-intercept	End Behaviour	Asymptotes*
Exponential Growth	$x \in R$	$y \in R, y > 0$	none	1	$x \to \infty, y \to \infty$	$y = 0$

increasing

*Asymptote is a straight line that a curve approaches but never touches.

$y = 3^x$

x	-1	0	1
y	$\frac{1}{3}$	1	3
	×2	×2	×2

$y = 2(3^x)$

$x \to \infty$

$y \to \infty$

2. **Paper-cutting Investigation: Exponential Decay *(Decreasing)***

(a) Take a large rectangular sheet of paper and cut it into halves. Put away the half you cut out. Record the number of cuts and the remaining portion of the original paper after the cut, as a fraction, in the table provided in part (c).

(b) Cut the paper into halves again. Put away the half you cut out and record the remaining portion as a fraction of the original paper.

(c) Continue cutting the remaining paper into halves and recording the number of cuts and the remaining portion of the original paper until you can no longer cut the paper. Record your findings in the following table.

Number of Cuts	Remaining Portion of Original Paper	Common Ratio $= \frac{1}{2}$
0	1	
1	$\frac{1}{2}$	$\frac{1}{2} \div 1 = \frac{1}{2}$
2	$\frac{1}{4}$	$\frac{1}{4} \div \frac{1}{2} = \frac{1}{2}$
3	$\frac{1}{8}$	$\frac{1}{8} \div \frac{1}{4} = \frac{1}{2}$
4	$\frac{1}{16}$	$\frac{1}{16} \div \frac{1}{8} = \frac{1}{2}$
5	$\frac{1}{32}$	$\frac{1}{32} \div \frac{1}{16} = \frac{1}{2}$

(d) Graph the relation, showing the number of cuts on the horizontal axis and the remaining portion as a fraction of the original paper on the vertical axis. (Graph on next page.)

Hint: you may want to set a bigger scale for your vertical axis.
For example: use 20 units as 1 to graph "fractions".

(e) If x represents the number of cuts and y represents the remaining portion of the original paper, write an equation for the relation. Decay (not growing) $y = \left(\frac{1}{2}\right)^x$

For exponential growth

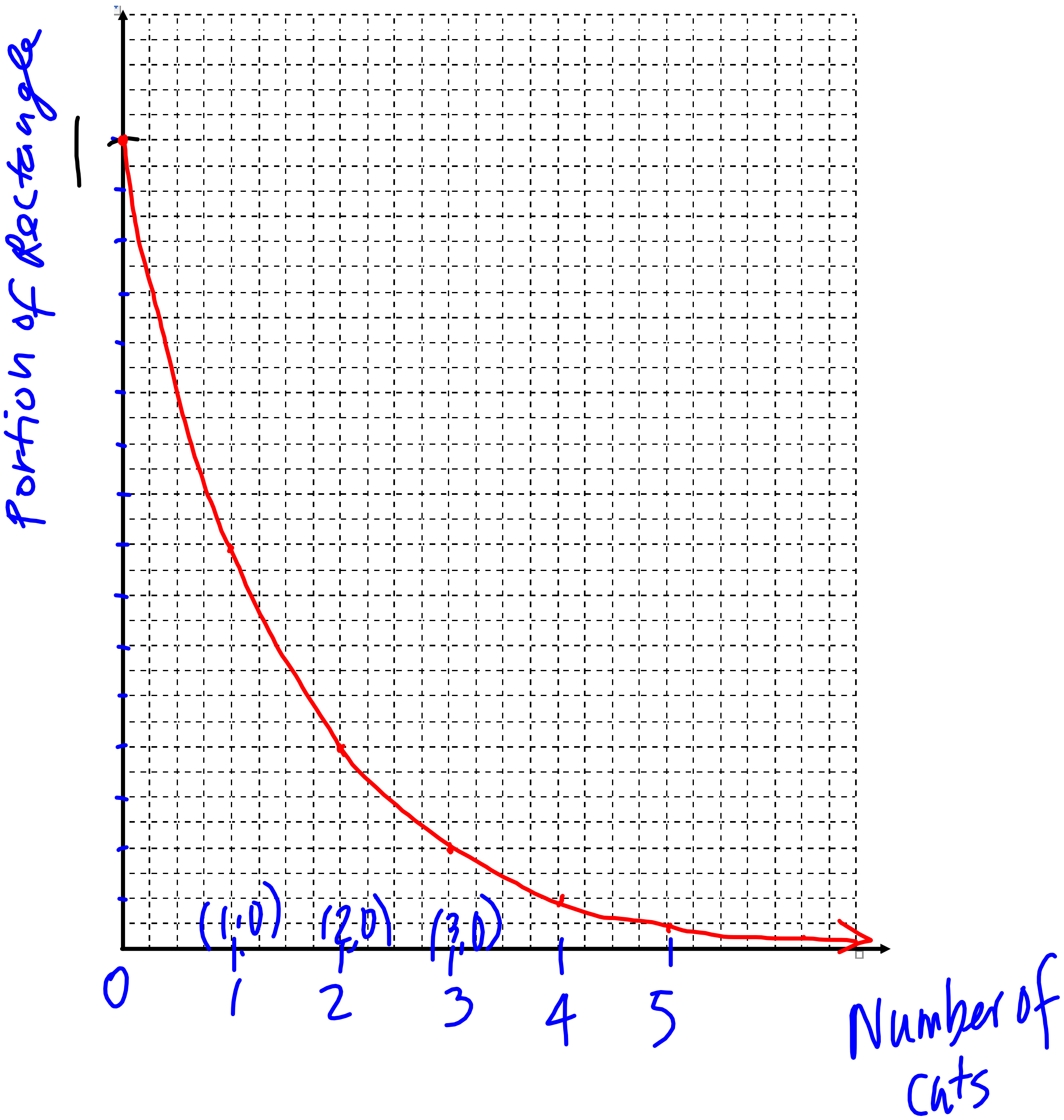
$y = b^x$, **where** $\boxed{0 < b < 1.}$

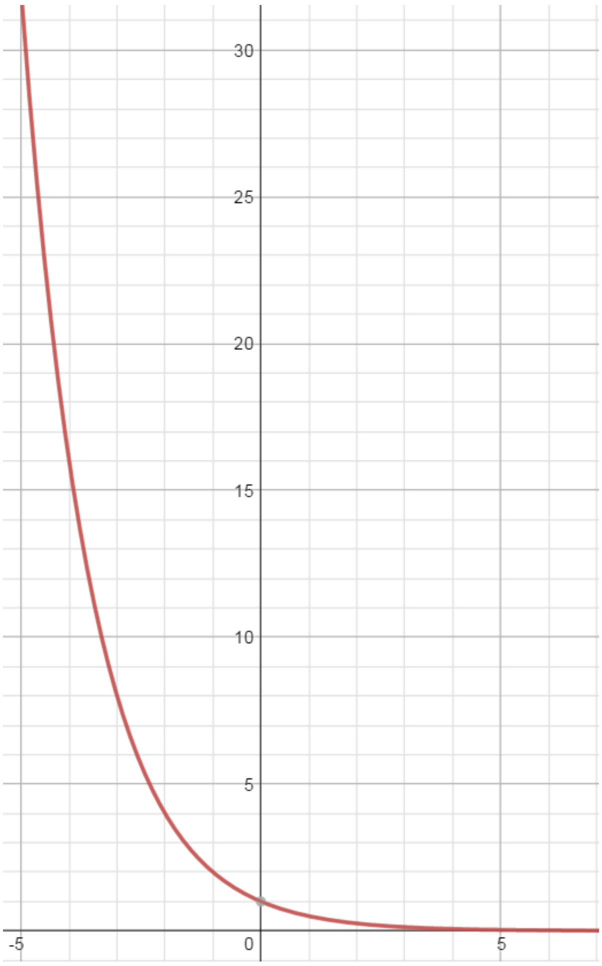
(f) Complete the following table for your graph.

Exponential Decay	Domain	Range	x-intercept	y-intercept	End Behaviour	Asymptotes*
	$x \in R$	$y \in R, y > 0$	none	1	$x \to \infty, y \to 0$	$y = 0$

*Asymptote is a straight line that a curve approaches but never touches.

decreasing and approaching 0


Portion of Rectangle
1
(1,0)
(2,0)
(3,0)
0
1
2
3
4
5
Number of cats

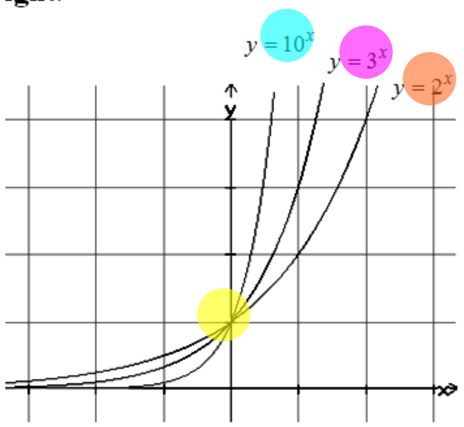


$x \to \infty$

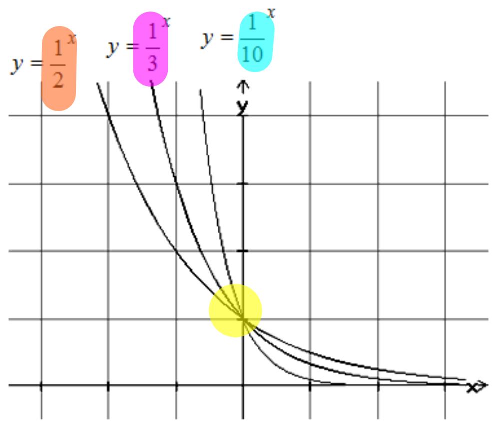
$y \to 0$

Properties of Exponential Relations $y = b^x$:

- A relation of the form $y = b^x$, where $b > 0$ and $b \neq 1$, is exponential.
- If $b > 1$, moving left to right, the graph increases very slowly for negative x-values and increases more rapidly for positive x-values. The graph is almost horizontal on the left and very steep on the right.



- If $0 < b < 1$, moving from left to right, the graph decreases very rapidly for negative x-values and decreases more slowly for positive x-values. The graph is almost horizontal on the right and very steep on the left.



- The y-intercept is 1.
- There is no x-intercept.
- The "growth" factor or "decay" factor is the base of the power, b, which is the common ratio between successive y-values.

Investigation: Exponential Relations $y = a(b)^x$, **where** a **is the initial amount or** y**-intercept**

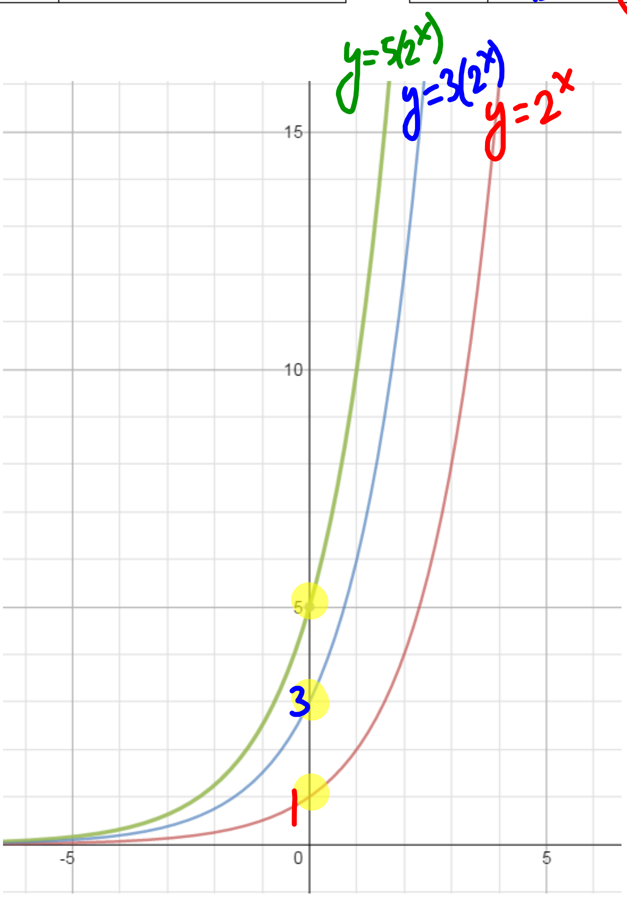
1. Graph $y = 2^x$, $y = 3(2^x)$, **and** $y = 5(2)^x$ **on the same axes and compare.**

a = vertical stretch factor

x	$a=1$ $y = 2^x$
-3	$2^{-3} = \frac{1}{8}$
-2	$2^{-2} = \frac{1}{4}$
-1	$2^{-1} = \frac{1}{2}$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$

x	$a=3$ $y = 3(2^x)$
-3	$3(2^{-3}) = 3(\frac{1}{8}) = \frac{3}{8}$
-2	$3(2^{-2}) = 3(\frac{1}{4}) = \frac{3}{4}$
-1	$3(2^{-1}) = 3(\frac{1}{2}) = \frac{3}{2}$
0	$3(2^0) = 3(1) = 3$
1	$3(2^1) = 3(2) = 6$
2	$3(2^2) = 3(4) = 12$
3	$3(2^3) = 3(8) = 24$

x	$a=5$ $y = 5(2)^x$
-3	$5(2^{-3}) = 5(\frac{1}{8}) = \frac{5}{8}$
-2	$5(2^{-2}) = 5(\frac{1}{4}) = \frac{5}{4}$
-1	$5(2^{-1}) = 5(\frac{1}{2}) = \frac{5}{2}$
0	$5(2^0) = 5(1) = 5$
1	$5(2^1) = 5(2) = 10$
2	$5(2^2) = 5(4) = 20$
3	$5(2^3) = 5(8) = 40$



All three graphs have different shapes. As b gets bigger, the curve gets narrower and closer to the y-axis.

All three graphs have different y-intercepts which are the same values as their respective a-values in their equations.

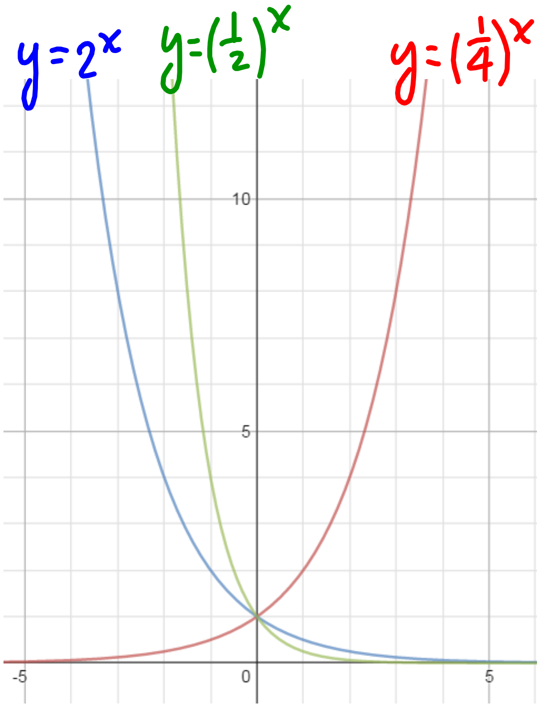
Investigation: Exponential Relations $y = b^x$, where x is multiplied or divided

2. Graph $y = 2^x$, $y = \left(\frac{1}{2}\right)^x$, and $y = \left(\frac{1}{4}\right)^x$ on the same axes and compare.

x	$y = 2^x$
-3	$2^{-3} = \frac{1}{8}$
-2	$2^{-2} = \frac{1}{4}$
-1	$2^{-1} = \frac{1}{2}$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$

x	$y = \left(\frac{1}{2}\right)^x$
-3	$\left(\frac{1}{2}\right)^{-3} = 2^3 = 8$
-2	$\left(\frac{1}{2}\right)^{-2} = 2^2 = 4$
-1	$\left(\frac{1}{2}\right)^{-1} = 2^1 = 2$
0	$\left(\frac{1}{2}\right)^0 = 1$
1	$\left(\frac{1}{2}\right)^1 = \frac{1}{2}$
2	$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$
3	$\left(\frac{1}{2}\right)^3 = \frac{1}{8}$

x	$y = \left(\frac{1}{4}\right)^x$
-3	$\left(\frac{1}{4}\right)^{-3} = 4^3 = 64$
-2	$\left(\frac{1}{4}\right)^{-2} = 4^2 = 16$
-1	$\left(\frac{1}{4}\right)^{-1} = 4^1 = 4$
0	$\left(\frac{1}{4}\right)^0 = 1$
1	$\left(\frac{1}{4}\right)^1 = \frac{1}{4}$
2	$\left(\frac{1}{4}\right)^2 = \frac{1}{16}$
3	$\left(\frac{1}{4}\right)^3 = \frac{1}{64}$



$y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$

have the same shape but they are reflection of each other in the y-axis. All three graphs have the same y-intercepts.

Pendulum Motion

A large pendulum was set in motion. With each complete swing, the pendulum's maximum distance from its rest position decreased. A motion sensor was used to obtain the data after every 5 swings, and a scatter plot of the data is given below.

Pendulum Motion

Number of Swings	Maximum Distance (cm)
0	23.5
5	19.9
10	17.0
15	14.6
20	10.4
25	8.7
30	7.6

Ratio

$19.9 \div 23.5 = 0.847$

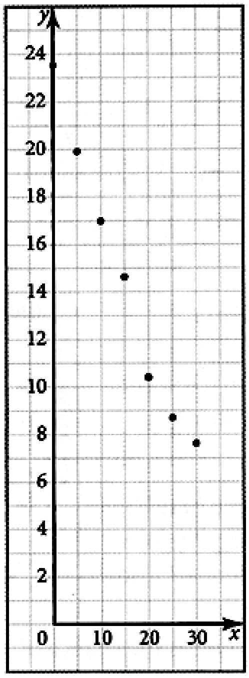
$17.0 \div 19.9 = 0.854$

$14.6 \div 17.0 = 0.859$

$10.4 \div 14.6 = 0.712$

$8.7 \div 10.4 = 0.837$

$7.6 \div 8.7 = 0.874$



(a) Describe the graph.

The graph looks like an exponential decay curve with a y-intercept of 24 cm and maximum distance decreasing and number of swings increasing.

(b) Calculate the ratio between successive distances. Is the relationship between the number of swings and the maximum distance of the pendulum swing exponential? Explain.

The relationship between the number of swings and the maximum distance of the pendulum swing can be considered as exponential. Though the ratio between successive distances is not constant, most ratios are consistently around 0.85 which can be considered as the common ratio.

(c) Is this relationship an example of *exponential growth* or *exponential decay*?

If this relationship is exponential, it will be an exponential decay since the maximum distance decreases as the number of swings increases. The function is decreasing.